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FACTOR ORBIT EQUIVALENCE OF COMPACT GROUP EXTENSIONS AND CLASSIFICATION OF FINITE EXTENSIONS OF ERGODIC AUTOMORPHISMS

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ABSTRACT

In §§1–5, we classify n-point extensions of ergodic automorphisms up to factor orbit-equivalence (which is the natural analogue of factor isomorphism). This classification is in terms of conjugacy classes of subgroups of the symmetric group on n points, and parallels D. Rudolph's classification of n-point extensions of Bernoulli shifts up to factor isomorphism. In §6, we give another proof of A. Fieldsteel's theorem on factor orbit-equivalence of compact group extensions.

§0. Foreword

Sections 1–5 of this paper are contained in the author's Ph.D. thesis [4], written under J. Feldman, and were circulated in a preprint titled *Classification of Finite Extensions of Ergodic Automorphisms up to Factor Orbit Equivalence.* The last section is based on a recent observation that the previous techniques extend to give another proof of A. Fieldsteel's theorem [3] on factor orbit-equivalence of compact group extensions.

§1. Introduction

Let S and S' be ergodic measure-preserving automorphisms of the Lebesgue spaces $(Y, \mathscr{C}, \nu)$ and $(Y', \mathscr{C}', \nu')$, respectively. (Throughout this paper, we identify sets and maps modulo sets of measure zero.) For $y \in Y$, we define

$$O_S(y) = \{S^n y : n \in \mathbf{Z}\}.$$

We say that S and S' are orbit equivalent if there exists a measure-preserving

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isomorphism $\Psi: (Y, \mathscr{C}, \nu) \to (Y', \mathscr{C}', \nu')$ such that for a.e. $y \in Y$,

$$\Psi(O_S(y)) = O_{S'}(\Psi(y)).$$

In 1958, H. Dye [2] proved that any two such S and S' are orbit equivalent.

Now let $\mathscr{B}$ be an S-invariant sub-sigma-field of $\mathscr{C}$ such that the quotient space is again a Lebesgue space, i.e., $T = S \mid \mathscr{B}$ is a "factor" of S. Similarly, let $T' = S' \mid \mathscr{B}'$ be a factor of S'. We consider the following question (posed by J. Feldman): Can Ψ be chosen to send $\mathscr{B}$ to $\mathscr{B}'$? If so, we say that (S, T) and (S', T') are "factor orbit-equivalent" (in analogy with a factor isomorphism, where Ψ would have to be an isomorphism between S and S' sending $\mathscr{B}$ to $\mathscr{B}'$). The ergodicity of S and S' implies that almost all atoms of $\mathscr{B}$ have the same cardinality, call it $n(\mathscr{B})$, and similarly for $\mathscr{B}'$. An obvious necessary condition is $n(\mathscr{B}) = n(\mathscr{B}')$. Is this sufficient? It turns out that the answer is no, even if the atoms are finite (also in all other cases). We obtain a classification of the inequivalent possibilities, in terms of conjugacy classes of transitive subgroups of the symmetric group $\mathscr{S}_n$ on n points, paralleling D. Rudolph's results [8] for the case when S and S' are Bernoulli shifts, where he got a classification up to factor isomorphism, rather than factor orbit-equivalence.

Now drop ergodicity of S, but still insist that T be ergodic. The classification has a natural extension to this case: the subgroups of $\mathscr{S}_n$ which are involved need no longer be transitive. (In the general case, the orbit equivalence Ψ between S and S' is only required to preserve the null sets, not necessarily the measures; in the ergodic case, this already implies that Ψ is measure-preserving.)

In order to carry out our classification, we will represent the system $(S, Y, \mathscr{C}, \mathscr{B}, \nu)$ with $n(\mathscr{B}) = n$ (finite) as a skew product in the following manner. We may assume (by making a factor isomorphism) that

$$(1) \quad \begin{cases} Y = X \times \{1, \dots, n\}, \\ \mathscr{B} = \mathscr{A} \times \{\{1, \dots, n\}, \phi\}, \\ \mathscr{C} = \mathscr{A} \times \mathscr{F}, \\ \nu = \mu \times p, \\ \text{and} \\ S(x, i) = (Tx, \sigma_x(i)), \qquad x \in X, \quad i = 1, \dots, n, \end{cases}$$

where $\mathscr{F}$ is the sigma-field generated by $\{1\}, \dots, \{n\}$, T is an ergodic measure-preserving automorphism of a Lebesgue space $(X, \mathscr{A}, \mu)$, $x \to \sigma_x$ is a measurable map from X to $\mathscr{S}_n$, and p is a probability measure on $\{1, \dots, n\}$ with $p(\{i\}) > 0$ for $i = 1, \dots, n$. If S is ergodic, then $p(\{i\}) = 1/n$ for $i = 1, \dots, n$.

Henceforth, we will consider systems $(S, Y, \mathscr{C}, \mathscr{B}, \nu)$, or simply (S, T), of the form given in (1). In this situation S is called a "n-point extension" of T. If S and

S' are two n-point extensions

$$S(x,i)=(Tx,\sigma_x(i))$$
$$S'(x',i)=(T'x',\sigma'_{x'}(i))$$

then (S,T) and (S',T') are factor orbit-equivalent iff there is an orbit equivalence $\phi: X\to X'$ of T and T' and a measurable map $x\to\tau_x$ from X into $\mathcal{S}_n$ such that for a.e. $x\in X$,

$$\tau_{Tx}\sigma_x\tau_x^{-1}=\sigma'_{(s_x,\phi(x))} \tag{2}$$

where the function $x\to s_x$ mapping $X\to\mathbf{Z}$ is such that for a.e. $x\in X$

$$\phi(Tx)=(T')^{s_x}(\phi(x)),$$

and for $x'\in X'$, $s\in\mathbf{Z}$ we define

$$\sigma'_{(s,x')}=\begin{cases}\sigma'_{(T')^{s-1}(x')}\cdots\sigma'_{T'x'}\sigma'_{x'}, & \text{if } s>0,\\ \mathrm{id}, & \text{if } s=0,\\ (\sigma'_{(T')^{s}(x')})^{-1}\cdots(\sigma'_{(T')^{-2}(x')})^{-1}(\sigma'_{(T')^{-1}(x')})^{-1}, & \text{if } s<0.\end{cases} \tag{3}$$

If ϕ is an orbit-equivalence of T and T' such that (2) holds, then the map$(x,i)\to(\phi(x),\tau_x(i))$ is a factor orbit-equivalence of (S,T) and (S',T'). Conversely, if $\Psi: X\times\{1,\dots,n\}\to X'\times\{1,\dots,n\}$ is a factor orbit equivalence of (S,T) and (S',T'), then Ψ must be of the form $\Psi(x,i)=(\phi(x),\tau_x(i))$ where $\phi: X\to X'$, $\tau_{(\cdot)}: X\to\mathcal{S}_n$, because Ψ must map $\mathcal{B}$ to $\mathcal{B}'$. It then follows from Ψ being an orbit equivalence of S and S' that ϕ is an orbit equivalence of T and T', and that (2) holds.

Similarly, (S,T) and (S',T') are factor isomorphic iff there is an isomorphism $\phi: X\to X'$ of T and T', and a measurable map $x\to\tau_x$ from X into $\mathcal{S}_n$ such that (2) holds and $p'(\{\tau_x(i)\})=p(\{i\})$ for a.e. $x\in X$, $i=1,\dots,n$. In this case we have $s_x=1$ a.e., which implies that $\sigma'_{(s_x,\phi(x))}=\sigma'_{\phi(x)}$ a.e. and (2) reduces to

$$\tau_{Tx}\sigma_x\tau_x^{-1}=\sigma'_{\phi(x)}\qquad\text{a.e.}$$

Our classification argument proceeds in three steps. First we define, for each subgroup G of $\mathcal{S}_n$, what we will call the "G-interchange property" for an n-point extension, and we show that for each n-point extension S of T there is a unique (up to conjugacy) G such that (S,T) is factor orbit-equivalent to an n-point extension S' of T' having the G-interchange property. Also, if (S,T) has the G-interchange property, and H is conjugate to G, then there is an (S',T') factor isomorphic to (S,T) having the H-interchange property. Next we construct, for each subgroup G of $\mathcal{S}_n$, an n-point extension having the G-interchange property. Finally, we make a slight extension of Dye's theorem to show that any two n-point extensions having the G-interchange property are factor orbit-equivalent. (In this argument we will assume some familiarity with the proof of Dye's theorem, as presented by W. Krieger [7].) Thus we will have a one-to-one correspondence between factor orbit-equivalence classes of n-point extensions of ergodic automorphisms and conjugacy classes of subgroups of $\mathcal{S}_n$.

K. Schmidt pointed out (in a conversation) that it can be seen from the Connes–Krieger classification result [1] (in the outer periodic case), that any two ergodic n-point extensions having the form given in (1), where each σ_x is a power of the cyclic permutation $(1\,2\cdots n)$, are factor orbit-equivalent. If we let G be the group generated by $(1\,2\cdots n)$, then in our classification, these n-point extensions correspond to the conjugacy class containing G.

Using methods quite different from both those in [1] and those in the present work, A. Fieldsteel [3] showed that if H is a compact metrizable group and S, S' are ergodic H-extensions of measure-preserving automorphisms T, T', respectively (i.e., S is defined on $X\times H$ by $S(x,h)=(Tx,\sigma(x)h)$, where $x\to\sigma(x)$ is a measurable function from X to H, and S' is defined similarly in terms of T' and another function $x'\to\sigma'(x')$ from X' to H), then (S,T) and (S',T') are factor orbit-equivalent. Fieldsteel used a metric on sequences of symbols introduced by D. Rudolph, and he gave an argument analogous to that of D. Ornstein's proof of the isomorphism theorem for Bernoulli shifts. In the present paper, we will give a proof of Fieldsteel's theorem using Dye's theorem methods.

§2. Setting up the correspondence between n-point extensions of ergodic automorphisms and conjugacy classes of subgroups of $\mathcal{S}_n$

Throughout this section, we will let S and S' denote n-point extensions of the ergodic automorphisms T and T':

$$S(x,i)=(Tx,\sigma_x(i))$$

$$S'(x',i)=(T'x',\sigma'_{x'}(i))$$

as in the introduction.

Let us recall the following.

Definition. Suppose R is a measure-preserving automorphism of a Lebesgue space and A, B are sets of positive measure. Then an invertible bi-measurable map $U\colon A\to B$ is an *R-isomorphism* if for a.e. $x\in A$, there is a $k_x\in\mathbf{Z}$ such that $Ux=R^{k_x}x$. (An R-isomorphism is necessarily measure-preserving.)

The first step in the proof of Dye's theorem, as presented in [7], is to observe that if R is an ergodic measure-preserving automorphism of a Lebesgue space, and A, B are sets with the same positive measure, then there is an R-isomorphism from A to B. A basic idea in our extension of Dye's theorem (to be carried out in §4) is to consider special T-isomorphisms which we define next. The above observation will then motivate the definition of the G-interchange property.

DEFINITIONS. Let $\sigma \in \mathscr{S}_n$, let $U: A \to B$ be a T-isomorphism, and let $x \to k_x$ map A to $\mathbf{Z}$ so that $Ux = T^{k_x}x$ for a.e. $x \in A$. Then U is a *T-σ-isomorphism* for the n-point extension S, if for a.e. $x \in A$, we have $\sigma = \sigma_{(k_x,x)}$, where for $x \in X$, $s \in \mathbf{Z}$, we define $\sigma_{(s,x)}$ as in (3) with the primes removed; or, equivalently,

$$S^{k_x}(x,i) = (T^{k_x}x, \sigma(i)) \qquad \text{for a.e. } x \in A, \quad i = 1, \dots, n.$$

For each subgroup G of $\mathscr{S}_n$, we say that (S, T) has the *G-interchange property* if

(i) for a.e. $x \in X$, $\sigma_x \in G$,

and

(ii) if $\sigma \in G$ and $A, B \in \mathscr{A}$ with $\mu(A) = \mu(B) > 0$, then there is a T-σ-isomorphism from A to B.

PROPOSITION. *For each n-point extension S of T, there is a subgroup G of $\mathscr{S}_n$, unique up to conjugacy, for which there is an n-point extension S' of T' such that (S', T') has the G-interchange property and is factor orbit-equivalent to (S, T). Furthermore, G is transitive iff S is ergodic.*

PROOF. As in [8], we begin by considering an ergodic component C of the "full extension" $\hat{S}$ of T defined on $(X \times \mathscr{S}_n, \mu \times q)$, where $q(\{\sigma\}) = 1/|\mathscr{S}_n|$, $\sigma \in \mathscr{S}_n$, by

$$\hat{S}(x, \sigma) = (Tx, \sigma_x \sigma).$$

Since T is ergodic, C must have the form

$$C = \bigcup_{j \in I} P_j \times \Gamma_j$$

where $\{P_j\}_{j \in I}$ is a partition of X, and the Γ_j's are subsets of $\mathscr{S}_n$ all having the same cardinality. In particular, each $\Gamma_j \neq \phi$; so for each $j \in I$, choose $\gamma_j \in \Gamma_j$. Let $j, k \in I$. Since $\hat{S} \mid C$ is ergodic, some power of $\hat{S}$ maps a subset of $P_j \times \{\gamma_j\}$ of positive measure to a subset of $P_k \times \{\gamma_k\}$. Hence there is some $\sigma \in \mathscr{S}_n$ such that $\sigma \Gamma_j = \Gamma_k$ and $\sigma \gamma_j = \gamma_k$, i.e., $\gamma_j^{-1}\Gamma_j = \gamma_k^{-1}\Gamma_k$. Thus there is a set $G \subset \mathscr{S}_n$ such that $\gamma_j^{-1}\Gamma_j = G$ for all $j \in I$. Then we have

$$C = \bigcup_{j \in I} P_j \times \gamma_j G.$$

The above argument shows that

$$\gamma_j^{-1}\Gamma_j = \bar{\gamma}_j^{-1}\Gamma_j, \qquad \text{for all } \bar{\gamma}_j \in \Gamma_j.$$

Hence

$$(\gamma_j^{-1}\bar{\gamma}_j)\gamma_j^{-1}\Gamma_j = \gamma_j^{-1}\Gamma_j \qquad \text{for all } \bar{\gamma}_j \in \Gamma_j,$$

i.e.

$$gG = G \qquad \text{for all } g \in G.$$

Therefore G is a group.

Let $\tau_x = \gamma_j^{-1}$ for $x \in P_j$, $j \in I$. Define another n-point extension S', by letting $T' = T$ and $\sigma'_x = \tau_{Tx}\sigma_x\tau_x^{-1}$. Then the map Ψ: $X \times \{1, \dots, n\} \to X \times \{1, \dots, n\}$ defined by $(x, i) \to (x, \tau_x(i))$ is a factor orbit-equivalence of (S, T) and (S', T). (Actually, it is a factor isomorphism if we choose $p'(\{i\}) = p(\{\tau_x^{-1}(i)\})$, which turns out to be constant for fixed i, as x varies.) Also, $\hat{S}$ and $\hat{S}'$ are isomorphic via $\hat{\phi}$: $X \times \mathcal{S}_n \to X \times \mathcal{S}_n$ defined by $(x, \sigma) \to (x, \tau_x\sigma)$.

We now verify that (S', T) has the G-interchange property. First observe that $\hat{\phi}(C)$ is an ergodic component of $\hat{S}'$ and

$$\hat{\phi}(C) = \hat{\phi}\left(\bigcup_{j \in I} P_j \times \gamma_j G\right) = \bigcup_{j \in I} (P_j \times \gamma_j^{-1}\gamma_j G) = X \times G.$$

Since $\hat{S}'$ leaves $X \times G$ invariant, for a.e. $x \in X$, $\hat{S}'(x, \mathrm{id}) = (Tx, \sigma'_x) \in X \times G$, i.e, $\sigma'_x \in G$. Now fix $\sigma \in G$ and let $A, B \in \mathcal{A}$ with $\mu(A) = \mu(B) > 0$. Then, because $\hat{S}' \,|\, X \times G$ is ergodic, there is an $\hat{S}'$-isomorphism mapping $A \times \{\mathrm{id}\}$ to $B \times \{\sigma\}$. This implies that there is a T-σ-isomorphism for S' mapping A to B. Hence (S', T) has the G-interchange property.

Note that if there is a T-σ-isomorphism from A to B and $\sigma(i) = j$, then there is an S'-isomorphism from $A \times \{i\}$ to $B \times \{j\}$. Hence, if G is transitive, the G-interchange property implies that S' is ergodic, and therefore S is ergodic. Conversely, if G preserves a proper subset Γ of $\mathcal{S}_n$, then $X \times \Gamma$ is a nontrivial invariant set for S', and thus S is not ergodic.

It remains to be shown that G is unique up to conjugacy. Suppose two factor orbit-equivalent n-point extensions S of T and S' of T' have the G- and H-interchange properties, respectively. Then there must be maps ϕ: $X \to X'$ and $\tau_{(\cdot)}$: $X \to \mathcal{S}_n$ so that (2) is satisfied. Let $Z \subset X$ and $\tau \in \mathcal{S}_n$ be such that $\mu(Z) > 0$ and $\tau_x = \tau$ for all $x \in Z$. We will show that $H = \tau G \tau^{-1}$. Let $g \in G$, and let $A, B \subset Z$ with $\mu(A) = \mu(B) > 0$. Let $A' = \phi(A)$, $B' = \phi(B)$. Since S has the G-interchange property, there exists a T-g-isomorphism U: $A \to B$ for S. Let U': $A' \to B'$ be defined by $U'(x') = \phi U \phi^{-1}(x')$ for $x' \in A'$. Then U' is a $T' - \tau g \tau^{-1}$-isomorphism for S'. Hence $\tau g \tau^{-1} \in H$, and since $g \in G$ was arbitrary, $\tau G \tau^{-1} \subset H$. By symmetry $\tau^{-1} H \tau \subset G$. Therefore $H = \tau G \tau^{-1}$. □

Proposition. *Suppose an n-point extension S of T has the G-interchange property and G is conjugate to H. Then there is an n-point extension S' of T such that (S', T) has the H-interchange property and is factor isomorphic to (S, T).*

PROOF. Let $H = \tau G \tau^{-1}$. Then the n-point extension S' of T determined by $\sigma'_x = \tau\sigma_x\tau^{-1}$ and $p'(\{i\}) = p(\{\tau^{-1}(i)\})$ has the required properties. □

§3. Construction of examples

Let $G = \{\sigma_1, \ldots, \sigma_m\}$ be a subgroup of $\mathcal{S}_n$. We now construct an n-point extension S of T having the G-interchange property. Let (T, X, μ) be a $(1/m, 1/m, \ldots, 1/m)$-Bernoulli shift, i.e. $X = \{1, \ldots, m\}^{\mathbf{Z}}$, $\mu = \Pi_{-\infty}^{\infty} q$ where q has the distribution $(1/m, 1/m, \ldots, 1/m)$ and if $x = (\ldots, x_{-1}, x_0, x_1, \ldots) \in X$, then $(Tx)_j = x_{j+1}$. Let $\sigma_x = \sigma_j$ for $x \in X$ such that $x_0 = j$. Let $S(x, i) = (Tx, \sigma_x(i))$, $x \in X$, $i = 1, \ldots, n$, and let $p(\{i\}) = 1/n$, $i = 1, \ldots, n$.

PROPOSITION. *The n-point extension S defined above has the G-interchange property.*

PROOF. Let $\sigma \in \mathcal{S}_n$, let $(c_j)_{-(l-1) \leqq j \leqq l-1}$, $(d_j)_{-(l-1) \leqq j \leqq l-1}$ be sequences of elements of $\{1, \ldots, m\}$, and let

$$C = \{x \in X \colon x_j = c_j,\ -(l-1) \leqq j \leqq l-1\},$$
$$D = \{x \in X \colon x_j = d_j,\ -(l-1) \leqq j \leqq l-1\}. \tag{4}$$

We will construct $C' \subset C$ and $D' \subset D$ with

$$\mu(C') = \mu(D') \geqq \frac{1}{2m}\,\mu(C)$$

and a $T - \sigma$-isomorphism $U \colon C' \to D'$ for S.

For each $k \in \mathbf{Z}$, let I_k be the sequence of integers of length $2l - 1$ given by

$$I_k = ((2k-1)l + 1, (2k-1)l + 2, \ldots, (2k+1)l - 1).$$

For $k > 0$, let C_k be the set of $x \in C$ such that:

(a) $(x_j)_{j \in I_k} = (d_{-(l-1)}, \ldots, d_{l-1})$,

(b) $(x_j)_{j \in I_h} \neq (c_{-(l-1)}, \ldots, c_{l-1})$, if $0 < h < k$,

(c) $(x_j)_{j \in I_h} \neq (d_{-(l-1)}, \ldots, d_{l-1})$, if $0 < h < k$,

(d) $\sigma_{x_t}\sigma_{x_{t-1}} \cdots \sigma_{x_2}\sigma_{x_1}\sigma_{x_0} = \sigma$ where $t = 2kl - 1$.

Clearly (a) and (c) imply that the C_k's, $k > 0$, are pairwise disjoint.

Let $C' = \bigcup_{k=1}^{\infty} C_k$, define U on C' by $U_x = T^{2kl_x}$ for $x \in C_k$, let $D_k = UC_k$ and let $D' = \bigcup_{k=1}^{\infty} D_k$. Since I_h shifted to the left by $2kl$ units is I_{h-k}, we have for $x \in D_k$,

$$(x_j)_{j\in I_{-k}} = (c_{-(l-1)}, \ldots, c_0, \ldots, c_{l-1}),$$

$$(x_j)_{j\in I_h} \neq (c_{-(l-1)}, \ldots, c_0, \ldots, c_{l-1}) \qquad \text{for } 0 < h < k,$$

and

$$(x_j)_{j\in I_0} = (d_{-(l-1)}, \ldots, d_0, \ldots, d_{l-1}).$$

Hence the D_k's, $k > 0$, are pairwise disjoint subsets of D. This, together with condition (d) in the definition of the C_k's, implies that $U\colon C' \to D'$ is a $T-\sigma$-isomorphism.

We now compute $\mu(C')$. Note that conditions (a), (b) and (c) in the definition of the C_k's do not place any constraints on $x_{(2k-1)l}$. Also, once we are given x_j, $0 \leqq j \leqq 2kl-1$, $j \neq (2k-1)l$, there is exactly one choice of $x_{(2k-1)l}$ to make (d) hold. Thus, condition (d) reduces the measure of C' by a factor $1/m$ of what it would be with only conditions (a), (b), and (c). Hence we have

$$\mu(C') = \begin{cases} \dfrac{1}{m}\,\mu(C), & \text{if } (c_{-(l-1)}, \ldots, c_{l-1}) = (d_{-(l-1)}, \ldots, d_{l-1}), \\ \\ \dfrac{1}{2m}\,\mu(C), & \text{otherwise.} \end{cases}$$

Now suppose A and B are subsets of X with $\mu(A) = \mu(B) > 0$. Take cylinder sets C and D of the form given in (4) such that

$$\mu(A \cap C) > \left(1 - \frac{1}{8m}\right)\mu(C),$$

$$\mu(B \cap D) > \left(1 - \frac{1}{8m}\right)\mu(D).$$

Let C' and D' and U be as constructed above. Then

$$\mu(A \cap C') > \tfrac{3}{4}\mu(C')$$

and

$$\mu(B \cap D') > \tfrac{3}{4}\mu(D') = \tfrac{3}{4}\mu(C').$$

Thus we have

$$\mu[(A \cap C') \cap U^{-1}(B \cap D')] > \tfrac{1}{2}\mu(C') > 0,$$

and

$$U \,\big|\, [(A \cap C') \cap U^{-1}(B \cap D')] \text{ is a } T-\sigma\text{-isomorphism.}$$

Then by Zorn's lemma (or even without it), it follows that there is a $T-\sigma$-isomorphism from A to B. □

§4. Completion of the classification result

The one-to-one correspondence between factor orbit-equivalence classes of n-point extensions of ergodic automorphisms will be established once we prove the following.

THEOREM. *Let G be a subgroup of $\mathcal{S}_n$. Let S and S' be n-point extensions of ergodic automorphisms T and T', respectively*:

$$S(x,i)=(Tx,\sigma_x(i)), \qquad x\in X, \quad i=1,\dots,n,$$

$$S'(x',i)=(T'x',\sigma'_{x'}(i)), \qquad x\in X', \quad i=1,\dots,n.$$

Then if (S,T) and (S',T') both have the G-interchange property, they are factor orbit-equivalent.

This theorem is proved by modifying the argument in [7], being careful that the constructed orbit equivalence ϕ between T and T' also makes the map Ψ: $X\times\{1,\dots,n\}\to X'\times\{1,\dots,n\}$ defined by $\Psi(x,i)=(\phi(x),i)$ for a.e. $x\in X$, $i=1,\dots,n$ a factor orbit-equivalence of (S,T) and (S',T'). For the convenience of the reader, we will repeat here some of the definitions given in [7], using essentially the same notation.

DEFINITIONS. A T-array α is a quadruple of the form

$$\alpha=(\Omega,A,A(\cdot),U(\cdot,\cdot)) \tag{5}$$

where Ω is a finite index set, $A\in\mathcal{A}$, $A(\cdot)$ is defined on Ω so that $\{A(\omega)\colon \omega\in\Omega\}$ is an ordered partition of A and $U(\cdot,\cdot)$ is defined on $\Omega\times\Omega$ so that

$$U(\omega,\omega')\colon A(\omega)\to A(\omega') \text{ is a } T\text{-isomorphism for } \omega,\omega'\in\Omega$$

and

$$U(\omega',\omega'')U(\omega,\omega')(x)=U(\omega,\omega'')(x) \qquad \text{for a.e. } x\in A(\omega), \quad \omega,\omega',\omega''\in\Omega.$$

For a T-array α in the form given in (5), we denote the partition $\{A(\omega)\colon \omega\in\Omega\}$ by $\mathcal{P}_\alpha$, and for $x\in A(\omega)$, we let $O_\alpha(x)=\{U(\omega,\omega')(x)\colon \omega'\in\Omega\}$.

Now suppose we are given a T-array α of the form in (5), and for some $\omega_0\in\Omega$ we are also given a T-array

$$\beta=(\Lambda,A(\omega_0),B(\cdot),V(\cdot,\cdot)).$$

Then the *refinement of α by means of β* is a T-array

$$\gamma = (\Omega \times \Lambda, A, C(\cdot), W(\cdot, \cdot))$$

where

$$C(\omega, \lambda) = U(\omega_0, \omega)B(\lambda), \qquad (\omega, \lambda) \in \Omega \times \Lambda$$

and

$$W((\omega, \lambda), (\omega', \lambda'))(x) = U(\omega_0, \omega')V(\lambda, \lambda')U(\omega, \omega_0)(x)$$

$$\text{for a.e. } x \in C(\omega, \lambda), \quad (\omega, \lambda), (\omega', \lambda') \in \Omega \times \Lambda.$$

For our argument here we will also need the following.

Definition. Let G be a subgroup of $\mathcal{S}_n$. Then a T-array $\alpha = (\Omega, A, A(\cdot), U(\cdot, \cdot))$ is a *T–G-array* for S if there is a function $r = r_\alpha : \Omega \times \Omega \rightarrow G$ such that for each $(\omega, \omega') \in \Omega \times \Omega$, $U(\omega, \omega')$ is a T–$r(\omega, \omega')$-isomorphism for S.

Note that if a function $r : \Omega \times \Omega \rightarrow G$ satisfies $r = r_\alpha$ for some T–G-array α with index set Ω, then

$$r(\omega', \omega'')r(\omega, \omega') = r(\omega, \omega''), \qquad \text{for } \omega, \omega', \omega'' \in \Omega. \tag{6}$$

Also observe that if $\alpha = (\Omega, A, A(\cdot), U(\cdot, \cdot))$ is a T–G-array and for some $\omega_0 \in \Omega$, $\beta = (\Lambda, A(\omega_0), B(\cdot), V(\cdot, \cdot))$ is also a T–G-array, then the refinement $\gamma = (\Omega \times \Lambda, A, C(\cdot), W(\cdot, \cdot))$ of α by means of β must also be a T–G-array, and we have

$$r_\gamma((\omega, \lambda), (\omega', \lambda')) = r_\alpha(\omega_0, \omega')r_\beta(\lambda, \lambda')r_\alpha(\omega, \omega_0), \quad \text{for } (\omega, \lambda), (\omega', \lambda') \in \Omega \times \Lambda.$$

We now proceed with a series of lemmas, similar to those in [7], which will lead to the proof of the above theorem. We assume throughout that G is a subgroup of $\mathcal{S}_n$, S, given by $S(x, i) = (Tx, \sigma_x(i))$, is an n-point extension of the ergodic automorphism T of $(X, \mathcal{A}, \mu)$, and S has the G-interchange property. We will omit the proofs of Lemmas 1 and 2, which are very easy, and just like the corresponding ones in [7].

Lemma 1. *Let $(\Omega, A, A(\cdot))$ be a partition of $A \in \mathcal{A}$ into sets of equal measure. Suppose $r : \Omega \times \Omega \rightarrow G$ satisfies* (6). *Then there is a T–G-array $\alpha = (\Omega, A, A(\cdot), U(\cdot, \cdot))$ such that $r = r_\alpha$.*

Lemma 2. *Let α be a T–G-array with index set $\{0, 1\}^N$, let $E \in \mathcal{A}$, and let $\varepsilon > 0$. Then there exists, for some $L \in \mathbf{N}$, a T–G-array β with index set $\{0, 1\}^{N+L}$*

which refines α and is such that

$$E \overset{\varepsilon}{\in} \mathscr{P}_\beta$$

(i.e. there is a union F of elements of $\mathscr{P}_\beta$ such that $\mu(E \triangle F) < \varepsilon$).

LEMMA 3. *Let $\varepsilon > 0$ and let $\alpha = (\{0,1\}^N, A, A(\cdot), U(\cdot,\cdot))$ be a T-array. Then for some $L \in \mathbf{N}$ there exists a T–G-array γ with index set $\{0,1\}^{N+L}$ such that for x in some subset of A of measure greater than $(1-\varepsilon)\mu(A)$, we have*

$$O_\alpha(x) \subset O_\gamma(x).$$

PROOF. Fix $\omega_0 \in \{0,1\}^N$. We can partition $A(\omega_0)$ into sets $F_1, \ldots, F_k$ such that for each $\omega \in \{0,1\}^N$ and each h, $1 \leqq h \leqq k$, there is some $\sigma = \sigma(h, \omega) \in G$ such that $U(\omega_0, \omega) \mid F_h$ is a $T-\sigma$-isomorphism. Then if L is sufficiently large, there is some partition $\{B(\lambda) : \lambda \in \{0,1\}^L\}$ of $A(\omega_0)$ such that each F_h, $1 \leqq h \leqq k$, contains a collection of sets in $\{B(\lambda) : \lambda \in \{0,1\}^L\}$ whose union covers all but a fraction less than ε of F_h. Then by Lemma 1, there is some T–G-array $\beta = (\{0,1\}^L, A(\omega_0), B(\cdot), V(\cdot,\cdot))$ Let $\Lambda_1 = \{\lambda \in \{0,1\}^L : B(\lambda) \subset F_h$, some $h = 1, \ldots, k\}$, and let $\Lambda_2 = \{0,1\}^L - \Lambda_1$. Then if $\lambda \in \Lambda_1$, $\omega \in \Omega$, $U(\omega_0, \omega) \mid B(\lambda)$ is a T–σ-isomorphism for some $\sigma = \sigma(\lambda_1, \omega) \in G$. Also we have

$$\mu\Big(\bigcup_{\lambda \in \Lambda_1} B(\lambda)\Big) > (1-\varepsilon)\mu(A(\omega_0))$$

and consequently

$$\mu\Big[\bigcup_{\omega \in \{0,1\}^N} \bigcup_{\lambda \in \Lambda_1} U(\omega_0, \omega) B(\lambda)\Big] > (1-\varepsilon)\mu(A).$$

We now define a new T-array $\bar\alpha = (\{0,1\}^N, A, A(\cdot), \bar U(\cdot,\cdot))$ by letting

$$\bar U(\omega_0, \omega) \mid B(\lambda) : B(\lambda) \to U(\omega_0, \omega) B(\lambda)$$

be defined for $\omega \in \{0,1\}^N - \{\omega_0\}$ by

$$\bar U(\omega_0, \omega) \mid B(\lambda) = U(\omega_0, \omega) \mid B(\lambda) \qquad \text{if } \lambda \in \Lambda_1$$

and

$$\bar U(\omega_0, \omega) \mid B(\lambda) \text{ is any } T\text{–}\sigma\text{-isomorphism} \qquad \text{for any } \sigma \in G, \text{ if } \lambda \in \Lambda_2.$$

(For $\omega, \omega' \in \{0,1\}^N$, $\bar U(\omega, \omega')$ must then be defined by $\bar U(\omega, \omega') = \bar U(\omega_0, \omega') \bar U(\omega_0, \omega)^{-1}$.) Let γ be the refinement of $\bar\alpha$ by means of β. Then γ is a T–G-array, and for $x \in \bigcup_{\omega \in \{0,1\}^N} \bigcup_{\lambda \in \Lambda_1} U(\omega_0, \omega) B(\lambda)$,

$$O_\alpha(x) = O_{\bar\alpha}(x) \subset O_\gamma(x). \qquad \square$$

LEMMA 4. *Let $\varepsilon > 0$ and let $\alpha = (\{0,1\}^N, X, A(\cdot), U(\cdot,\cdot))$ be a $T-G$-array. Then there exists some $L \in \mathbf{N}$ and a T–G-array γ with index set $\{0,1\}^{N+L}$ which refines α and is such that*

$$\mu\{x \in X:\ Tx \in O_\gamma(x)\} > 1 - \varepsilon.$$

PROOF. Note that if T–G-arrays in this lemma are replaced by T-arrays, then we get lemma 6 in [7], which we will use together with our Lemma 3 in the proof here. By lemma 6 in [7], there is an $M_1 \in \mathbf{N}$ and a T-array δ with index set $\{0,1\}^{N+M_1}$ which refines α and is such that

$$\mu\{x \in X:\ Tx \in O_\delta(x)\} > 1 - \varepsilon/2.$$

Now fix $\omega_0 \in \{0,1\}^N$ and consider the restriction of δ to $A(\omega_0)$ to be a T-array with index set $\{0,1\}^{M_1}$. Then, by Lemma 3, there exists an $M_2 \in \mathbf{N}$ and a $T-G$-array β on $A(\omega_0)$ with index set $\{0,1\}^{M_1+M_2}$ such that

$$O_\delta(x) \cap A(\omega_0) \subset O_\beta(x)$$

for all x in some subset of $A(\omega_0)$ of measure greater than $(1-\varepsilon/2^{N+1})\times \mu(A(\omega_0))$.

We take γ to be the T–G-array obtained by refining α by means of β. Then γ has index set $\{0,1\}^{N+L}$, where $L = M_1 + M_2$ and

$$\mu\{x \in X:\ Tx \in O_\gamma(x)\} \geqq \mu[\{x \in X:\ Tx \in O_\delta(x)\} \cap \{x:\ O_\delta(x) \subset O_\gamma(x)\}]$$
$$> 1 - \varepsilon. \qquad \square$$

PROOF OF THEOREM. This proof is very much like that in [7], but here we will have to be slightly more careful, because we are not working with a covenient prototype (as the adding machine is in [7]).

Let $(E_k)_{k\in\mathbf{N}}$ and $(E'_k)_{k\in\mathbf{N}}$ be generating sequences of elements of $\mathcal{A}$ and $\mathcal{A}'$, respectively, in which each term appears infinitely often. We now produce inductively sequences $(\alpha_l)_{l\in\mathbf{N}}$ and $(\alpha'_l)_{l\in\mathbf{N}}$ of T–G-arrays (for S) and T'–G-arrays (for S'), respectively, of the form

$$\alpha_l = (\{0,1\}^{N(l)}, X, A_l(\cdot), U_l(\cdot,\cdot))$$

and

$$\alpha'_l = (\{0,1\}^{N(l)}, X', A'_l(\cdot), U'_l(\cdot,\cdot))$$

having the following properties:

(7)
(i) $\alpha_{l+1}[\alpha'_{l+1}]$ is a refinement of $\alpha_l[\alpha'_l]$, $l \in \mathbf{N}$,
(ii) $r_{\alpha_l} = r_{\alpha'_l}$, $l \in \mathbf{N}$,
(iii) $E_k \overset{1/k}{\in} \mathcal{P}_{\alpha_{4k-3}}$, $k \in \mathbf{N}$,
(iv) $\mu\{x \in X\colon Tx \in O_{\alpha_{4k-2}}(x)\} > 1 - 1/k$, $k \in \mathbf{N}$,
(v) $E'_k \overset{1/k}{\in} \mathcal{P}_{\alpha'_{4k-1}}$, $k \in \mathbf{N}$,
(vi) $\mu\{x' \in X'\colon T'x' \in O_{\alpha_{4k}}(x')\} > 1 - 1/k$, $k \in \mathbf{N}$.

Suppose $k \geqq 0$ and $\alpha_1, \alpha_2, \ldots, \alpha_{4k}$, $\alpha'_1, \alpha'_2, \ldots, \alpha'_{4k}$ have been constructed to satisfy the above properties. We then apply Lemmas 2 and 4 on the space $(X, \mathcal{A}, \mu)$ to obtain $\alpha_{4(k+1)-3}$ refining α_{4k} such that (iii) holds for $k+1$ and to obtain $\alpha_{4(k+1)-2}$ refining $\alpha_{4(k+1)-3}$ such that (iv) holds for $k+1$. Next we apply Lemma 1 to copy over these refinements of α_{4k} to the space $(X', \mathcal{A}', \mu')$ to obtain $\alpha'_{4(k+1)-3}$ refining α'_{4k}, and $\alpha'_{4(k+1)-2}$ refining $\alpha'_{4(k+1)-3}$ such that (ii) holds for $l = 4(k+1)-3$ and $4(k+1)-2$. Then we reverse the roles of $(X, \mathcal{A}, \mu)$ and $(X', \mathcal{A}', \mu')$, applying Lemmas 2 and 4 to obtain $\alpha'_{4(k+1)-1}$ refining $\alpha'_{4(k+1)-2}$ and $\alpha'_{4(k+1)}$ refining $\alpha'_{4(k+1)-1}$ such that (v) and (vi) hold for $k+1$ and then applying Lemma 1 to copy these refinements over to $(X, \mathcal{A}, \mu)$ to obtain $\alpha_{4(k+1)-1}$ refining $\alpha_{4(k+1)-2}$ and $\alpha_{4(k+1)}$ refining $\alpha_{4(k+1)-1}$ such that (ii) holds for $l = 4(k+1)-1$ and $4(k+1)$. Thus we obtain sequences of arrays $(\alpha_l)_{l\in\mathbf{N}}$, $(\alpha'_l)_{l\in\mathbf{N}}$ having all the properties listed above.

We now indicate how to get an orbit equivalence ϕ between T and T' which makes the map $\psi = X \times \{1, \ldots, n\} \to X' \times \{1, \ldots, n\}$ defined by

(8) $$\psi(x, i) = (\phi(x), i) \qquad \text{for a.e. } x \in X, \quad i = 1, \ldots, n$$

a factor orbit-equivalence between (S, T) and (S', T'). For each $A \in \mathcal{A}$, let $[A]$ denote the equivalence class containing A in the Boolean sigma-algebra $\mathcal{A}/\mathcal{N}$ of measurable sets modulo null sets in $\mathcal{A}$, and similarly for $A' \in \mathcal{A}'$. Let d be the metric on $\mathcal{A}/\mathcal{N}$ defined by $d([A],[B]) = \mu(A \triangle B)$, and define d' similarly on $\mathcal{A}'/\mathcal{N}'$. Then $(\mathcal{A}/\mathcal{N}, d)$ and $(\mathcal{A}'/\mathcal{N}', d')$ are complete metric spaces. Thus there is an isometry Φ from $\mathcal{A}/\mathcal{N}$ onto $\mathcal{A}'/\mathcal{N}'$ preserving all the Boolean operations (i.e. Φ is a measure-preserving set isomorphism) determined by setting

$$\Phi([A_l(\omega)]) = [A'_l(\omega)] \qquad \text{for } \omega \in \{0,1\}^{N(l)}, \quad l \in \mathbf{N}.$$

Then there is a measure-preserving point isomorphism $\phi\colon X \to X'$ which induces the set isomorphism Φ (see [9], page 582), i.e. $[\phi(A)] = \Phi([A])$ for $A \in \mathcal{A}$.

Fix $l \in \mathbf{N}$ and $\omega_1, \omega_2 \in \{0,1\}^{N(l)}$. We will show that $\phi^{-1}U'_l(\omega_1, \omega_2)\phi = U_l(\omega_1, \omega_2)$ a.e. on $A_l(\omega_1)$. Let $m > l$ and write $\{0,1\}^{N(m)} = \{0,1\}^{N(l)} \times \{0,1\}^M$. Let

$\lambda \in \{0,1\}^M$. Then

$$\begin{aligned}(\phi^{-1}U'_l(\omega_1,\omega_2)\phi)A_m(\omega_1,\lambda) &= \phi^{-1}U'_l(\omega_1,\omega_2)A'_m(\omega_1,\lambda)\\ &= \phi^{-1}A'_m(\omega_2,\lambda)\\ &= A_m(\omega_2,\lambda)\\ &= U_l(\omega_1,\omega_2)A_m(\omega_1,\lambda),\end{aligned}$$

with all of these equalities holding up to a set of measure zero. Since the $A_m(\omega_1,\lambda)$'s generate (as m and λ vary) the restriction of $\mathcal{A}$ to $A_l(\omega_1)$, this implies that $(\phi^{-1}U'_l(\omega_1,\omega_2)\phi)(x) = U_l(\omega_1,\omega_2)(x)$ for a.e. $x \in A_l(\omega_1)$. Thus, for a.e. $x \in X$,

$$U'_l(\omega_1,\omega_2)\phi(x) = \phi(U_l(\omega_1,\omega_2)x) \tag{9}$$

for $l \in \mathbf{N}$, $\omega_1,\omega_2 \in \{0,1\}^{N(l)}$, where ω_1 is such that $x \in A_l(\omega_1)$. By (7iv), for a.e. $x \in X$,

$$O_T(x) = \{U_l(\omega_1,\omega_2)x : l \in \mathbf{N}, \omega_1,\omega_2 \in \{0,1\}^{N(l)}, \text{ where } \omega_1 \text{ is such that } x \in A_l(\omega_1)\}.$$

Since each α_l is a T–G-array, this implies that for a.e. $x \in X$, $i = 1,\dots,n$,

$$O_S(x,i) = \{(U_l(\omega_1,\omega_2)x, r_{\alpha l}(\omega_1,\omega_2)(i)) : l \in \mathbf{N}, \omega_1,\omega_2 \in \{0,1\}^{N(l)}, \text{ where } \omega_1 \text{ is such that } x \in A_l(\omega_1)\}.$$

A similar statement holds for $O_{S'}(x',i)$. From this representation of $O_S(x,i)$, $O_{S'}(x',i)$, (7ii) and (9), it follows that for the map ψ, defined as in (8), we have

$$\Psi(O_S(x,i)) = O_{S'}(\Psi(x,i)) \qquad \text{for a.e. } x \in X, \quad i = 1,\dots,n. \qquad \square$$

§5. Some further remarks

Let us now consider the case of an uncountable extension of an ergodic measure-preserving automorphism T of a Lebesgue space $(X,\mathcal{A},\mu)$. Let $(\bar{X},\bar{\mathcal{A}},\bar{\mu})$ be another Lebesgue space, and let $Y = X \times \bar{X}$, $\mathcal{B} = \mathcal{A} \times \bar{X}$, $\mathcal{C} = \mathcal{A} \times \bar{\mathcal{A}}$, $\nu = \mu \times \bar{\mu}$. Let S be an ergodic measure-preserving automorphism of $(Y,\mathcal{C},\nu)$ preserving $\mathcal{B}$ of the form $S(x,\bar{x}) = (Tx, \mathcal{T}_x(\bar{x}))$ for $x \in X$, $\bar{x} \in \bar{X}$ where $(x,\bar{x}) \to \mathcal{T}_x(\bar{x})$ is jointly measurable, and for a.e. $x \in X$, $\mathcal{T}_x$ is a measure-preserving automorphism of $(\bar{X},\bar{A},\bar{\mu})$. Suppose S' and T' are of a form similar to that of S and T. Then $(S,\mathcal{C},\mathcal{B})$ and $(S',\mathcal{C}',\mathcal{B}')$, or simply (S,T) and (S',T'), are factor orbit-equivalent if there exists a map ψ: $X \times \bar{X} \to X' \times \bar{X}'$ of the form

$$\psi(x,\bar{x}) = (\phi(x), \tau_x(\bar{x})) \tag{10}$$

where $\phi: X \to X'$ is an orbit equivalence of T and T', for a.e. $x \in X$, $\tau_x: \bar{X} \to \bar{X}'$ is a measure-preserving isomorphism, and for a.e. $x \in X$,

$$\tau_{Tx}\mathcal{T}_x\tau_x^{-1} = \mathcal{T}'_{(s_x,\phi(x))} \qquad \bar{\mu}\text{-a.e.},$$

where the function $x \to s_x$ mapping $X \to \mathbf{Z}$ is such that for a.e. $x \in X$,

$$\phi(Tx) = (T')^{s_x}(\phi(x)),$$

and for $x' \in X'$, $s \in \mathbf{Z}$, we define

$$\mathcal{T}'_{(s,x')} = \begin{cases} \mathcal{T}'_{(T')^{s-1}(x')} \cdots \mathcal{T}'_{T'x'}\mathcal{T}'_{x'}, & \text{if } s > 0, \\ \text{id}, & \text{if } s = 0, \\ (\mathcal{T}'_{(T')^{s}(x')})^{-1} \cdots (\mathcal{T}'_{(T')^{-2}(x')})^{-1}(\mathcal{T}'_{(T')^{-1}x'})^{-1}, & \text{if } s < 0. \end{cases} \tag{11}$$

We now define a property in the same spirit as the G-interchange property.

Definition. A set $C \in \mathcal{C}$ will be called a *preservable strip of width* α for some α, $0 \leqq \alpha \leqq 1$, if for a.e. $x \in X$, $C_x = C \cap (\{x\} \times \bar{X})$ has $\bar{\mu}$-measure α and if $A, B \in \mathcal{A}$ with $\mu(A) = \mu(B) > 0$, then there exists a T-isomorphism U: $A \to B$ such that if $x \to k_x$ is defined by $Ux = T^{k_x}x$ for a.e. $x \in A$, then for a.e. $x \in A$, $\mathcal{T}_{(k_x,x)}(C_x) = C_{Ux}$ up to a set of $\bar{\mu}$-measure zero, where we define $\mathcal{T}_{(s,x)}$ as in (11) with the primes removed; or, equivalently, $S^{k_x}(C_x) = C_{Ux}$ up to a set of $\bar{\mu}$-measure zero. A *nontrivial preservable strip* is one of width α, where $0 < \alpha < 1$.

It is easy to see that preservable strips get mapped to preservable strips under any factor orbit equivalence. Hence having a preservable strip of width α is an invariant of factor orbit equivalence.

We now look at some examples.

Proposition. *Suppose T is an ergodic measure-preserving automorphism of $(X, \mathcal{A}, \mu)$, $\bar{T}$ is a Bernoulli shift on $(\bar{X}, \bar{\mathcal{A}}, \bar{\mu})$ and $S = T \times \bar{T}$. Then (S, T) does not have a nontrivial preservable strip.*

Proof. Suppose $0 < \alpha < 1$ and $C \subset X \times \bar{X}$ is a preservable strip of width α. Let R be the equivalence relation $x \sim y$ if there exists some $k \in \mathbf{Z}$ such that $T^k x = y$ and $S^k C_x = C_y$ up to a set of $\bar{\mu}$-measure zero. Then R is hyperfinite. (This is easy to see if we use the definition of a hyperfinite equivalence relation as being the union of an increasing sequence of finite equivalence relations.) Thus there exists some measure-preserving automorphism $\hat{T}$ of $(X, \mathcal{A}, \mu)$ such that for a.e. $x \in X$, $\{y: (x, y) \in R\} = O_{\hat{T}}(x)$. From the definition of C being preservable, it follows that $\hat{T}$ is ergodic. Define a function mapping X to $\mathbf{Z}$ by $x \to k_x$, where

$\hat{T}x = T^{k_x}x$. Since $\hat{T}$ is ergodic, for a.e. $x \in X$, we have $\hat{T}^i x \neq \hat{T}^j x$ if $i, j \in \mathbf{Z}$, $i \neq j$, which implies that the sequence $f_x^1, f_x^2, \dots$ defined by

$$f_x^n = k_x + k_{\hat{T}x} + \cdots + k_{\hat{T}^{n-1}x} \tag{12}$$

consists of distinct integers. Let $\hat{S}(x, \bar{x}) = (\hat{T}x, \bar{T}^{k_x}\bar{x})$. Then C is a $\mu \times \bar{\mu}$-a.e. invariant set for $\hat{S}$. We will reach a contradiction by establishing the following (where for convenience we write S and T instead of $\hat{S}$ and $\hat{T}$).

LEMMA. *Suppose T is an ergodic measure-preserving automorphism of $(X, \mathcal{A}, \mu)$, $\bar{T}$ is a Bernoulli shift on $(\bar{X}, \bar{\mathcal{A}}, \bar{\mu})$, and $x \to k_x$ is a measurable function from X to $\mathbf{Z}$ such that if f_x^n is defined as in* (12), *for a.e. $x \in X$, zero appears in the sequence $f_x^1, f_x^2, \dots$ only finitely many times. Then the transformation S on $(X \times \bar{X}, \mathcal{A} \times \bar{\mathcal{A}}, \mu \times \bar{\mu})$ defined by $S(x, \bar{x}) = (Tx, \bar{T}^{k_x}\bar{x})$ is ergodic.*

PROOF. Let $C, D \subset \bar{X}$ be cylinder sets (for the Bernoulli shift $\bar{T}$) based on coordinates $-l, \dots, 0, \dots, l$, and let $A, B \in \mathcal{A}$. We will establish the ergodicity of S by showing that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (\mu \times \bar{\mu})[(A \times C) \cap S^n(B \times D)] = \mu(A)\mu(B)\bar{\mu}(C)\bar{\mu}(D). \tag{13}$$

For each n, we have

$$(A \times C) \cap S^n(B \times D) = \bigcup_{x \in T^{-n}A \cap B} [\{T^n x\} \times (C \cap \bar{T}^{f_x^n} D)].$$

Also, for a.e. $x \in X$, any fixed integer occurs only finitely often in the sequence $f_x^1, f_x^2, \dots$. Thus, given $\varepsilon > 0$, there is some K such that for each x in some set $G \subset X$ with $\mu(G) > 1 - \varepsilon$, $|f_x^n| > 2l$ for each $n \geqq K$. Now if $|f_x^n| > 2l$, $\bar{\mu}(C \cap \bar{T}^{f_x^n} D) = \bar{\mu}(C)\bar{\mu}(D)$. Hence for $n \geqq K$, we have

$$\begin{aligned}(\mu \times \bar{\mu})&[(A \times C) \cap S^n(B \times D)] \\ &= \bar{\mu}(C)\bar{\mu}(D)\mu(T^{-n}A \cap B \cap G) + \int_{T^{-n}A \cap B \cap (X - G)} \bar{\mu}(C \cap \bar{T}^{f_x^n} D)\, d\mu(x).\end{aligned}$$

Thus, for $n \geqq K$,

$$|(\mu \times \bar{\mu})[(A \times C) \cap S^n(B \times D)] - \bar{\mu}(C)\bar{\mu}(D)\mu(T^{-n}A \cap B)| < \varepsilon.$$

Since T is ergodic,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$

Thus, for N large, $(1/N)\Sigma_{n=1}^{N}(\mu\times\bar{\mu})[(A\times C)\cap S^n(B\times D)]$ is within 2ε of $\mu(A)\mu(B)\bar{\mu}(C)\bar{\mu}(D)$. Since ε was arbitrary, (13) follows. □

REMARK. The same argument shows that if T is assumed to be weakly mixing then S is weakly mixing; if T is mixing, then S is mixing. In the ergodic and weakly mixing cases, it suffices to assume that for a.e. $x\in X$, zero appears with density zero in the sequence $f_x^1, f_x^2, \dots$.

The above system (S, T) can now be used to construct examples with very special types of preservable strips. For convenience in describing these examples, we will replace $\bar{X}$ by $\bar{X}\times\{1,\dots,n\}$ in the set-up given at the beginning of this section.

PROPOSITION. *Let T be an ergodic measure-preserving automorphism of $(X,\mathcal{A},\mu)$, $\bar{T}$ a Bernoulli shift on $(\bar{X},\bar{\mathcal{A}},\bar{\mu})$, $x\to\sigma_x$ a measurable map from X to $\mathcal{S}_n$. Let $\mathcal{F}$ be the sigma-field on $\{1,\dots,n\}$ consisting of all subsets of $\{1,\dots,n\}$ and let p be the measure $p(\{i\})=1/n$ for $i=1,\dots,n$. Define S on $(X\times(\bar{X}\times\{1,\dots,n\}),\ \mathcal{A}\times(\bar{\mathcal{A}}\times\mathcal{F}),\ \mu\times(\bar{\mu}\times p))$ by*

$$S(x,(\bar{x},i))=(Tx,(\bar{T}\bar{x},\sigma_x(i))). \tag{14}$$

Then any preservable strip C for (S,T) is of width m/n for some $m=0,1,\dots,n$, with μ-a.e. C_x being equal (up to a set of $(\bar{\mu}\times p)$-measure zero) to a union of m of the sets $\bar{X}\times\{1\},\bar{X}\times\{2\},\dots,\bar{X}\times\{n\}$ (which depends on x).

PROOF. Let C be a nontrivial preservable strip for (S,T) and define R, $\hat{T}$ and $x\to k_x$ as in the proof of the preceding proposition. Let $\hat{\sigma}_x=\sigma_{(k_x,x)}$, where $\sigma_{(s,x)}$ is defined as in (3), with the primes removed. Then C is a $\mu\times(\bar{\mu}\times p)$-a.e. invariant set for the transformation $\hat{S}$ on $X\times(\bar{X}\times\{1,\dots,n\})$ defined by

$$\hat{S}(x,(\bar{x},i))=(\hat{T}x,(\bar{T}^{k_x}\bar{x},\hat{\sigma}_x(i))).$$

Now consider the finite extension of $\hat{T}$ defined on $(X\times\{1,\dots,n\},\mathcal{A}\times\mathcal{F},\mu\times p)$ by

$$(x,i)\to(\hat{T}x,\hat{\sigma}_x(i)). \tag{15}$$

Let $D\subset X\times\{1,\dots,n\}$ be defined by

$$D=\{(x,i):\ \bar{\mu}[C_x\cap(\bar{X}\times\{i\})]>0\}.$$

Then D is invariant under the map given in (15). Thus, by the ergodicity of $\hat{T}$, there is some $m\in\{1,\dots,n\}$ such that

$$\operatorname{card}\{i\in\{1,\dots,n\}:(x,i)\in D\}=m \qquad \text{for } \mu\text{-a.e. } x,$$

and we can write D as a disjoint union of sets $D_1,\dots,D_m$, where each D_j is contained in an ergodic component for the map in (15), and each D_j satisfies

$$\operatorname{card}\{i\in\{1,\dots,n\}: (x,i)\in D_j\}=1 \qquad \text{for } \mu\text{-a.e. } x.$$

Let $C_j = C\cap \tilde{D}_j$, $j=1,\dots,m$, where

$$\tilde{D}_j=\{(x,(\bar{x},i)): (x,i)\in D_j, \bar{x}\in\bar{X}\}.$$

Because D_j is contained in an ergodic component for the map in (15), it follows that C_j is itself a preservable strip for (S,T). Let $E_j\subset X\times\bar{X}$ be given by

$$E_j=\{(x,\bar{x}): (x,(\bar{x},i))\in C_j \text{ for some } i=1,\dots,n\}.$$

Then E_j is a preservable strip for $(T\times\bar{T},T)$. Hence, by the preceding proposition, E_j must be $(\mu\times\bar{\mu})$-a.e. equal to $X\times\bar{X}$. Therefore we have $C_j=\tilde{D}_j$ $\mu\times(\bar{\mu}\times p)$-a.e. □

Proposition. *Let (S,T) be as in the preceding proposition. Then $X\times(\bar{X}\times\{1,\dots,n\})$ can be covered by pairwise disjoint preservable sets for (S,T) of width $1/n$.*

Proof. The space $X\times\{1,\dots,n\}$ can be covered by pairwise disjoint sets $D_1,\dots,D_n$ such that each D_j is contained in an ergodic component for the transformation on $X\times\{1,\dots,n\}$ given by $(x,i)\to(Tx,\sigma_x(i))$, and for a.e. x and each $j=1,\dots,n$, $\operatorname{card}\{i: (x,i)\in D_j\}=1$. For $j=1,\dots,n$, let $C_j=\{(x,(\bar{x},i)): (x,i)\in D_j, \bar{x}\in\bar{X}\}$. Then $C_1,\dots,C_n$ are pairwise disjoint and each C_j is a preservable strip for (S,T) of width $1/n$. □

The existence of preservable strips of width $1/n$ and no preservable strips of widths other than multiples of $1/n$ for the systems (S,T) considered in the last two propositions already gives (as we let n vary) infinitely many factor orbit-equivalence classes. But note also that if (S,T) and (S',T') are of the form given in (14) (with the same n, but possibly different $x\to\sigma_x$ and $x'\to\sigma'_{x'}$ and possibly different Bernoulli shifts $\bar{T}$ and $\bar{T}'$) then any factor orbit-equivalence between (S,T) and (S',T') must be of the form given in (10), with $\bar{X}$ and $\bar{X}'$ replaced by $\bar{X}\times\{1,\dots,n\}$ and $\bar{X}'\times\{1,\dots,n\}$, respectively, where for a.e. $x\in X$, there is some $\gamma_x\in\mathcal{S}_n$ such that for $i=1,\dots,n$,

$$\tau_x(\bar{X}\times\{i\})=\bar{X}'\times\{\gamma_x(i)\} \qquad (\mu'\times p)\text{-a.e.}$$

Therefore, a necessary condition for (S,T) and (S',T') of the form given in (14) to be factor orbit-equivalent is that the corresponding n-point extensions $(x,i)\to(Tx,\sigma_x(i))$ and $(x',i)\to(T'x,\sigma'_{x'}(i))$ be factor orbit-equivalent.

§6. Factor orbit equivalence of compact group extensions

Let H be a compact metrizable group, and let T be an ergodic measure-preserving automorphism of a Lebesgue space $(X, \mathscr{A}, \mu)$. Then an *H-extension of T* is an automorphism S of $Y = X \times H$ of the form $S(x, h) = (Tx, \sigma(x)h)$, where $\sigma: X \to H$ is measurable.

THEOREM [3]. *Let H be a compact metrizable group with Haar measure ρ, and let S and S' be H-extensions of T and T' which are ergodic with respect to $\mu \times \rho$ and $\mu' \times \rho$, respectively. Then (S, T) and (S', T') are factor orbit equivalent.*

Fix a right-invariant metric d on H, and let $\mathscr{B}(h; \varepsilon)$ denote an open ball of radius ε about h in H.

DEFINITION. Let $N \subset H$. Let $A, B \in \mathscr{A}$ and let $U: A \to B$ be a T-isomorphism. Then U is a $T-N$-isomorphism for the H-extension S if for a.e. $x \in A$, we have $\sigma(k_x, x) \in N$, where k_x satisfies $U(x) = T^{k_x}x$, and for $x \in X$, $s \in \mathbf{Z}$, we define $\sigma(s, x)$ as in (3) in §1.

DEFINITION. Let $\alpha = (\Omega, A, A(\cdot), U(\cdot, \cdot))$ be a T-array, as defined in §4. Let $\omega_0 \in \Omega$, $r: \Omega \backslash \{\omega_0\} \to H$, $\varepsilon = \varepsilon(\omega) > 0$ for $\omega \in \Omega \backslash \{\omega_0\}$. Then α is a $T-H$-$(r, \omega_0, \varepsilon)$-array if for each $\omega \in \Omega \backslash \{\omega_0\}$, $U(\omega_0, \omega)$ is a $T - \mathscr{B}(r(\omega); \varepsilon(\omega))$-isomorphism.

LEMMA 1. *Let $\bar{\varepsilon} > 0$ and let $(\Omega, A, A(\cdot))$ be a partition of $A \in \mathscr{A}$ into sets of equal measure. Let $\omega_0 \in \Omega$ and $r: \Omega \backslash \{\omega_0\} \to H$. Then there is a $T - H$-$(r, \omega_0, \varepsilon)$-array $\alpha = (\Omega, A, A(\cdot), U(\cdot, \cdot))$, where $\varepsilon = \varepsilon(\omega) = \bar{\varepsilon}$ for all $\omega \in \Omega \backslash \{\omega_0\}$.*

LEMMA 2. *Let $\alpha = (\Omega, X, A(\cdot), U(\cdot, \cdot))$ be a T-array. Assume $\Omega = \{0, 1\}^N$. Let $\bar{\varepsilon} > 0$ and let $E \in \mathscr{A}$. Then there exists a T-array $\beta = (\Lambda, A(\omega_0), B(\cdot), V(\cdot, \cdot))$, $\Lambda = \{0, 1\}^L$, such that the refinement $\gamma = (\Omega \times \Lambda, X, \mathscr{C}(\cdot), W(\cdot, \cdot))$ of α by means of β satisfies the following conditions:*

(i) $\mu\{x \in X: Tx \in \mathcal{O}_\gamma(x)\} > 1 - \bar{\varepsilon}$,

(ii) $E \overset{\bar{\varepsilon}}{\in} P_\gamma$ and

(iii) if we fix a choice of $\lambda_0 \in \Lambda$, then for each $\lambda \in \Lambda \backslash \{\lambda_0\}$, $V(\lambda_0, \lambda)$ is a $T - \mathscr{B}(\bar{r}(\lambda), \bar{\varepsilon})$-isomorphism for some $\bar{r}(\lambda) \in H$.

LEMMA 3. *Let $\alpha = (\Omega, X, A(\cdot), U(\cdot, \cdot))$ be a $T - H - (r, \omega_0, \varepsilon)$-array. Assume $\Omega = \{0, 1\}^N$. Let $\bar{\varepsilon} > 0$. Then there exists a T-array $\beta = (\Lambda, A(\omega_0), B(\cdot), V(\cdot, \cdot))$, $\Lambda = \{0, 1\}^L$, such that the refinement $\gamma = (\Omega \times \Lambda, X, \mathscr{C}(\cdot), W(\cdot, \cdot))$ of α by means of β satisfies the following conditions:*

(i) There exists a subset $\Lambda_1 \subset \Lambda$ with $\operatorname{card} \Lambda_1 > (1 - \bar{\varepsilon}) \operatorname{card} \Lambda$ *such that for*

$\lambda \in \Lambda_1$, $\omega \in \Omega$, $U(\omega_0, \omega) \,|\, B(\lambda)$ is a $T - \mathcal{B}(\bar{r}(\omega, \lambda), \bar{\varepsilon})$-isomorphism for some $\bar{r}(\omega, \lambda) \in H$ with $d(\bar{r}(\omega, \lambda), r(\omega)) < \varepsilon(\omega)$, and

(ii) *For all $\lambda, \lambda' \in \Lambda$, $V(\lambda, \lambda')$ is a $T - \mathcal{B}(\mathrm{id}_H, \bar{\varepsilon})$-isomorphism.*

PROOF OF THEOREM. Let $(E_k)_{k=1,2,\ldots}$ and $(E'_k)_{k=1,2,\ldots}$ be generating sequences of elements of $\mathcal{A}$ and $\mathcal{A}'$, respectively, in which each term appears infinitely often. Let η_k satisfy $0 < \eta_k < 1$ and $\sum_{k=1}^{\infty} \eta_k < \infty$. Let ε_k be such that $0 < \varepsilon_k < \eta_k$ and if $g, g_1, h, h_1 \in H$ with $d(g, g_1) < \varepsilon_k$, $d(h, h_1) < \varepsilon_k$ then $d(gh^{-1}, g_1 h_1^{-1}) < \eta_k$. We will define inductively sequences $(\alpha_l)_{l=1,2,\ldots}$ and $(\alpha'_l)_{l=1,2,\ldots}$ of T-arrays and T'-arrays, respectively of the form

$$\alpha_l = (\{0,1\}^{N(l)}, X, A_l(\cdot), U_l(\cdot, \cdot))$$

and

$$\alpha'_l = (\{0,1\}^{N(l)}, X', A'_l(\cdot), U'_l(\cdot, \cdot))$$

such that $\alpha_{l+1}[\alpha'_{l+1}]$ is a refinement of $\alpha_l[\alpha'_l]$, $l = 1, 2, \ldots$.

For each $l = 1, 2, \ldots$, we have one index $\omega_{0,l} \in \{0,1\}^{N(l)}$ such that $A_{l+1}(\omega_{0,l+1}) \subset A_l(\omega_{0,l})$ and $A'_{l+1}(\omega_{0,l+1}) \subset A'_l(\omega_{0,l})$. Also, for $l = 1, 2, \ldots, \alpha_l[\alpha'_l]$ is a $T-H$-$(r_l[r'_l], \omega_{0,l}, \delta_l)$-array, where $r_l[r'_l]\colon \{0,1\}^{N(l)} \backslash \{\omega_{0,l}\} \to H$ and $\delta_l = \delta_l(\omega) > 0$ for $\omega \in \{0,1\}^{N(l)} \backslash \{\omega_{0,l}\}$. We let $r_l(\omega_{0,l}) = \mathrm{id}_H$, $r'_l(\omega_{0,l}) = \mathrm{id}_H$, and $\delta_l(\omega_{0,l}) = 0$. We will carry out a construction such that for all ω in a subset of $\{0,1\}^{N(l)}$ whose cardinality is greater than $(1 - \tilde{\delta}_l)$ of $\mathrm{card}(\{0,1\}^{N(l)})$, $\delta_l(\omega) < \tilde{\delta}_l$, where

$$\tilde{\delta}_l = \begin{cases} \varepsilon_k/3, & \text{if } l = 3k-2, \\ 2\varepsilon_k/3, & \text{if } l = 3k-1, \\ \varepsilon_k, & \text{if } l = 3k, \end{cases}$$

$k = 1, 2, \ldots$.

For $l = 1, 2, \ldots$, define $\theta_l\colon X \to H$ by

$$\theta_l(x) = r'_l(\omega_{0,l}, \omega) r_l(\omega_{0,l}, \omega)^{-1}, \qquad x \in A_l(\omega).$$

We alternatingly refine the α_l and α'_l arrays using Lemmas 2 and 3. Assume the arrays have been constructed for $l = 3k-3$. (We may take α_0 and α'_0 to be trivial arrays.) To go to the $l = 3k-2$ stage, we apply Lemma 3 with $\bar{\varepsilon}$ such that $\bar{\varepsilon} < \varepsilon_k/3$, and $d(g, \mathrm{id}_H) < \bar{\varepsilon}$ and $d(h, h_0) < \bar{\varepsilon}$ imply $d(gh, h_0) < \varepsilon_k/3$. Let $\Omega = \{0,1\}^{N(3k-3)}$, $\omega_0 = \omega_{0,3k-3}$, $\{0,1\}^{N(3k-2)} = \Omega \times \Lambda$, $\omega_{0,3k-2} = (\omega_0, \lambda_0)$ for some $\lambda_0 \in \Lambda_1$,

$$r_{3k-2}(\omega, \lambda) = \begin{cases} \bar{r}(\omega, \lambda) & \text{for } \omega \in \Omega \backslash \{\omega_0\}, \quad \lambda \in \Lambda_1, \\ r_{3k-3}(\omega) & \text{for } \omega \in \Omega \backslash \{\omega_0\}, \quad \lambda \notin \Lambda_1, \\ \mathrm{id}_H & \text{for } \omega = \omega_0, \quad \lambda \in \Lambda, \end{cases}$$

and similarly for the "prime" space. By labeling appropriately, we may use the same λ_0 and Λ_1, but different $\bar{r}$'s, for the construction of α_{3k-2} and α'_{3k-2}. Then $\delta_{3k-2}(\omega,\lambda)<\varepsilon_k/3$ except for a set of (ω,λ) of cardinality less than $\varepsilon_k/3$ of $\mathrm{card}(\Omega\times\Lambda)$. Also $d(r_{3k-2}(\omega,\lambda), r_{3k-2}(\omega))<\delta_{3k-3}(\omega)$ for all $(\omega,\lambda)\in\Omega\times\Lambda$, and similarly for r'. Consequently, $d(\theta_{3k-2}(x),\theta_{3k-3}(x))<\eta_{k-1}$ except for x in a set of measure less than ε_{k-1}.

We now use Lemma 2 to get α_{3k-1} from α_{3k-2} with $\mu\{x\in X\colon Tx\in\mathcal{O}_{\alpha_{3k-1}}(x)\}> 1-1/k$, $E_k \overset{1/k}{\subset} \mathcal{P}_{\alpha_{3k-1}}$ and condition (iii) of Lemma 2 is satisfied with $\bar{\varepsilon}$ chosen so that $\bar{\varepsilon}<\varepsilon_k/3$ and $d(h,h_0)<\bar{\varepsilon}$ imply that $d(gh,g_0h_0)<2\varepsilon_k/3$. Let $\Omega=\{0,1\}^{N(3k-2)}$, $\omega_0=\omega_{0,3k-2}$, $\{0,1\}^{N(3k-1)}=\Omega\times\Lambda$, $\omega_{0,3k-1}=(\omega_0,\lambda_0)$, and $r_{3k-1}(\omega,\lambda)= r_{3k-2}(\omega)\bar{r}(\lambda)$ for all $(\omega,\lambda)\in\Omega\times\Lambda$, where we let $\bar{r}(\lambda_0)=\mathrm{id}_H$. The α'_{3k-1} array is produced by refining α'_{3k-2} so that (iii) (but not (i) and (ii)) of Lemma 2 holds with the same $\bar{r}(\lambda)$ function. (This is possible by Lemma 1.) Then $r_{3k-1}(\omega,\lambda)= r'_{3k-2}(\omega)\bar{r}(\lambda)$, $\delta_{3k-1}(\omega,\lambda)<2\varepsilon_k/3$ except for ω in a subset of cardinality less than $\varepsilon_k/3$ of $\mathrm{card}\,\Omega$, and $\theta_{3k-1}(x)=\theta_{3k-2}(x)$ for all x.

Finally, we reverse the roles of the "prime" and "nonprime" arrays in the preceding step. We apply Lemma 2 to get α'_{3k} (analogously to α_{3k-1} above) with $\bar{\varepsilon}$ chosen so that $\bar{\varepsilon}<\varepsilon_k/3$, and $d(g,g_0)<2\varepsilon_k/3$ and $d(h,h_0)<\bar{\varepsilon}$ imply that $d(gh,g_0h_0)<\varepsilon_k$. Then apply Lemma 1 to get α_{3k} (analogously to α'_{3k-1}). Then $\delta_{3k}(\omega,\lambda)<\varepsilon_k$ except for ω in a subset of cardinality less than $\varepsilon_k/3$ of $\mathrm{card}\,\Omega$, and $\theta_{3k}(x)=\theta_{3k-1}(x)$ for all x.

Clearly $\theta_l(x)$ converges a.e. to some $\theta(x)$. Let $\varphi\colon X\to X'$ be the orbit equivalence between T and T' arising from the α_l, α'_l arrays (as in §4). Then $(x,g)\to(\varphi x,\theta(x)\cdot g)$ is an orbit equivalence between S and S'. □

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